A Discrete Theory of Irregular Sampling*

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ABSTRACT

We derive algorithms for the iterative reconstruction of discrete band-limited signals from their irregular samples. They converge at a geometric rate, and all constants are computable explicitly. We also treat the related problem of interpolation of trigonometric polynomials and give estimates for the condition number of certain positive definite Toeplitz matrices.

INTRODUCTION

In signal analysis one faces frequently the following problem: given the samples $s(t_n)$, $n \in \mathbb{Z}$, of a signal $s(t) \in L^2(\mathbb{R})$, try to find s. If s is band-limited, i.e., if s does not contain high frequencies, and if the sampling sequence is dense enough, then s is uniquely determined, and the sampling is stable, i.e., the reconstruction $s(t_n) \to s$ is a continuous map from l^2 into $L^2(\mathbb{R})$.

The reconstruction from equally spaced samples through the cardinal series is a standard tool in signal analysis; see e.g. [20]. On the other hand, the problem for irregular sampling sequences is more involved and leads to difficult mathematical questions; see [7] for an outline of the problem and references. The emphasis of recent contributions to irregular sampling has

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been on computable solutions and the derivation of reconstruction algorithms; see [7, 8, 12, 19, 24].

All these algorithms deal with continuous signals on $\mathbb{R}$ and are still a step away from the numerical implementation.

In this paper we give a direct treatment of the discrete problem of irregular sampling. This approach has the advantage that we analyze the algorithms in the exact version in which they are implemented, and not in an intermediate form. We shall give sufficient conditions for the convergence of the iterative algorithms and explicit error estimates. Since the discrete problem of irregular sampling is finite-dimensional and linear, a reconstruction amounts to solving a large overdetermined system of linear equations or to inverting a large matrix. The literature offers a variety of algorithms for the iterative solution of large linear systems, e.g. [23, 14]. Therefore the task is to decide which of these algorithms is appropriate for the discrete irregular sampling problem. Our results are twofold:

(1) We show that Richardson's method is applicable, and we obtain explicit estimates for the rate of convergence and give a range for the relaxation parameter (Theorem 2).

(2) We show how the original reconstruction problem can be reduced to a lower-dimensional Toeplitz system. In particular, we derive an estimate for the condition number of this system (Theorem 5). As a consequence the direct reconstruction of the signal by the inversion of the Toeplitz matrix of the system becomes feasible.

The inspiration for these reconstruction methods came from the sampling theory of band-limited functions on $\mathbb{R}$ [7, 12]. We hope that some of the ideas and techniques presented are also of interest in linear algebra.

Numerical experiments with these algorithms and comparison with other reconstruction methods have been carried out by H. G. Feichtinger [3, 4, 9]. So far the results with the iterative algorithms of Theorem 2 and 5 have been very convincing.

The finite-dimensional model of irregular sampling is closely related to the solution of Vandermonde systems and to the interpolation of trigonometric polynomials (see Sections 1.1 and 3). For these two problems also, efficient direct solution methods are known; cf. [2, 10, 15, 21]. In this paper, however, we emphasize iterative methods and the related quantitative aspects such as rates of convergence and condition numbers.

The paper is organized in the following way. In the first section we explain the discrete model of irregular sampling, give a precise formulation to the problem, and review some required inequalities. Section 2 contains the main results, namely, three iterative algorithms for the reconstruction of a discrete band-limited signal from irregular samples. In Section 3 we touch briefly on the related problem of interpolation of trigonometric polynomials, and Section 4 provides an equivalent formulation which involves Toeplitz matrices.

1. DISCRETE BAND-LIMITED SIGNALS

1.1. Statement of the Problem

In digital signal processing a signal consists of a finite number of data, i.e., a sequence of numbers $s(n)$, $n = 0, \dots, N-1$, where N is possibly very large. In practice s is sampled at a subset $0 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N-1$. The problem of irregular sampling asks under which conditions s can be completely recovered from $s(n_i)$.

To treat this problem in analogy to sampling of band-limited functions, we make the following model. The index set $\{0, 1, \dots, N-1\}$ is identified with the finite cyclic group $\mathbb{Z}_N$, and thus all signals are understood as periodic sequences with period N, i.e. $s(j) = s(j + lN)$ for $j = 0, 1, \dots, N-1$, $l \in \mathbb{Z}$. The l^2 norm

$$\|s\| = \left(\sum_{n=0}^{N-1} |s(n)|^2 \right)^{1/2} \tag{1}$$

represents the energy of s.

On $\mathbb{Z}_N$ the discrete Fourier transform (DFT) is defined by

$$\hat{s}(k) = N^{-1/2} \sum_{n=0}^{N-1} s(n) e^{-2\pi i k n/N}. \tag{2}$$

Then $s \mapsto \hat{s}$ is a unitary operator on $l^2(\mathbb{Z}_N)$ and $\|s\| = \|\hat{s}\|$.

Band-limited functions on $\mathbb{Z}_N$ are defined similarly to band-limited functions on $\mathbb{R}$. Using the periodicity of s and $\hat{s}$, we admit negative indices and define for $0 < M < N/2$ the space of discrete band-limited signals of bandwidth M by

$$\mathscr{B}_M = \left\{ s(n) \in l^2(\mathbb{Z}_N) \,\middle|\, \hat{s}(k) = 0 \text{ for } |k| > M \right\}. \tag{3}$$

The orthogonal projection P of $l^2(\mathbb{Z}_N)$ onto B_M is given by $(Ps)\hat{}\,(k) = \hat{s}(k)$ for $|k| \leqslant M$ and $(Ps)\hat{}\,(k) = 0$ for $M < |k| \leqslant N/2$.

In many applications it is reasonable to assume that no high frequencies occur and that the given signal is band-limited.

The problem of irregular sampling of discrete band-limited signals can now be stated in the following way. Given a sequence $0 \leqslant n_1 < n_2 < \cdots < n_r < N$ and the samples $s(n_i)$, $i = 1, \dots, r$, of a signal $s \in \mathscr{B}_M$, is s uniquely determined by its samples? If so, what is a practical method to reconstruct s?

The reconstruction from equally spaced samples is standard [15, 20]. Assume that $n_i = di$, $i = 0, 1, \dots, r = (N/d) - 1$ for some d dividing N and $d \leqslant N/(2M + 1)$, then

$$s(n) = \sum_{i=0}^{r} s(di) \frac{d \sin[(2M + 1)\pi(n - di)/N]}{N \sin[\pi(n - di)/N]}, \tag{4}$$

which is the discrete version of the classical Shannon-Whittaker-Kotel'nikov sampling theorem.

The reconstruction from irregular samples poses more problems. Given the sampling values $s(n_i)$ of $s \in \mathscr{B}_M$ on some sampling set $\{n_i\}$, the reconstruction amounts to solving the following system of linear equations:

$$N^{-1/2} \sum_{k=-M}^{M} \hat{s}(k) e^{2\pi i k n_j / N} = s(n_j), \qquad j = 1, \dots, r, \tag{5}$$

for $\hat{s}(k)$. As the $r \times (2M + 1)$ matrix $a_{jk} := e^{2\pi i k n_j / N}$ is a Vandermonde matrix, it has always full rank and (5) always has a unique solution under the hypothesis $r \geqslant 2M + 1$ and $s \in \mathscr{B}_M$.

Retaining only $2M + 1$ equations of (5), one can solve this system directly in $O(M^2)$ operations [2]. However, the removal of redundant samples tends to make the system ill conditioned. We will therefore seek an algorithm which makes use of all the given information and is well conditioned.

1.2. *Some Properties of Discrete Signals*

While band-limited functions on $\mathbb{R}$ are entire functions of exponential type, discrete band-limited signals also have some smoothness that distinguishes them from arbitrary signals on $\mathbb{Z}_N$. Indeed, a version of Bernstein's inequality is valid.

LEMMA 1 (Bernstein's inequality). *Let Δs denote the difference sequence $\Delta s(n) = s(n + 1) - s(n)$ (and by periodicity $\Delta s(N - 1) = s(0) - s(N - 1)$). Then for all $s \in \mathscr{B}_M$*

$$\|\Delta s\| \leqslant 2 \sin \frac{\pi M}{N} \|s\|. \tag{6}$$

Proof. Denote $w = e^{2\pi i/N}$, and observe that for $s \in \mathscr{B}_M$

$$(\Delta s)\hat{}\,(k) = N^{-1/2} \sum_{n=0}^{N-1} [s(n+1) - s(n)]\overline{w}^{kn} = \hat{s}(k)(w^k - 1). \quad (7)$$

Since $s \mapsto \hat{s}$ is unitary and $\hat{s}(k) = 0$ for $M < |k| \leqslant N/2$, one derives

$$\|\Delta s\| = \left(\sum_{k=-M}^{M} |\hat{s}(k)|^2 |w^k - 1|^2 \right)^{1/2}$$

$$\leqslant \max_{|k| \leqslant M} |w^k - 1| \left(\sum_{k=-M}^{M} |\hat{s}(k)|^2 \right)^{1/2} \leqslant 2 \sin \frac{\pi M}{N} \|s\| \quad (8)$$

since for $|k| \leqslant M$ we have $|w^k - 1| = |e^{2\pi i k/N} - 1| = 2\sin(\pi k/N) \leqslant 2\sin(\pi M/N)$. ■

Conversely, it is always possible to control the norm of a signal through the norm of its difference sequence. This is the content of the discrete version of Wirtinger's inequality [18].

LEMMA 2 (Discrete Wirtinger inequality).

(1) *Assume that* $s(1) = 0$. *Then for* $d > 0$ *we have*

$$\sum_{n=1}^{d} |s(n)|^2 \leqslant \left(4 \sin^2 \frac{\pi}{2(2d-1)} \right)^{-1} \sum_{n=1}^{d-1} |\Delta s(n)|^2. \quad (9)$$

(2) *If* $s(0) = s(d+1) = 0$ *and* $\Delta^2 s(n) = s(n+2) - 2s(n+1) + s(n)$, *then*

$$\sum_{n=1}^{d} |s(n)|^2 \leqslant \left(16 \sin^4 \frac{\pi}{2(d+1)} \right)^{-1} \sum_{n=1}^{d-1} |\Delta^2 s(n)|^2. \quad (10)$$

In both inequalities the constants are sharp.

1.3. *Iterative Reconstruction*

The following simple lemma contains the essence of many important iterative reconstruction algorithms. It is derived from the well-known inversion of a linear operator by means of a Neumann series. If A is a matrix, then the lemma is essentially Richardson's method for the iterative solution of the linear system $Af = h$ [14, 23].

LEMMA 3. *Let A be a bounded operator on a Banach space $(B, \|\cdot\|_B)$ such that*

$$\|f - Af\|_B \leqslant \gamma \|f\|_B \qquad \text{for all} \quad f \in B \tag{11}$$

and for a fixed constant γ, $0 \leqslant \gamma < 1$.

Then f can be reconstructed from Af by means of the iteration

$$f_0 = Af \quad \text{and} \quad f_{n+1} = f_n + A(f - f_n), \tag{12}$$

and

$$f = \lim_{n \mapsto \infty} f_n \qquad \text{in } B$$

with the error estimate

$$\|f - f_n\|_B \leqslant \gamma^{n+1} \|f\|_B. \tag{13}$$

To avoid unnecessary repetitions, we say that the algorithm (12) *converges to f geometrically with rate γ.*

Proof. The operator norm of $\mathrm{Id} - A$ on B satisfies $\|\mathrm{Id} - A\|_{\mathrm{Op}} \leqslant \gamma < 1$; therefore A is invertible and $A^{-1} = \sum_{n=0}^{\infty} (\mathrm{Id} - A)^n$ with $\|A^{-1}\|_{\mathrm{Op}} \leqslant (1 - \gamma)^{-1}$. Now we show by induction that

$$f_n = \sum_{k=0}^{n} (\mathrm{Id} - A)^k Af. \tag{14}$$

This is clear for $n = 0$ from (12), and follows for $n > 0$ from $f_{n+1} = Af + (\mathrm{Id} - A)\sum_{k=0}^{n} (\mathrm{Id} - A)^k Af$. Since $\sum_{k=n+1}^{\infty} (\mathrm{Id} - A)^k A = (\mathrm{Id} - A)^{n+1}$, the error estimate now follows from

$$\|f - f_n\|_B = \|(\mathrm{Id} - A)^{n+1} f\|_B \leqslant \gamma^{n+1} \|f\|_B. \qquad \blacksquare \tag{15}$$

REMARKS.

(a) For an iterative reconstruction of f with a geometric rate of convergence it is therefore sufficient

(1) to find an approximation operator A that requires only the given data on f as its input, and
(2) to prove an estimate $\|f - Af\|_B \leqslant \gamma \|f\|_B$.

(b) Once the convergence of the basic iteration (12) is established, the algorithm can often be accelerated. For instance, if A is a positive definite matrix or a positive, invertible operator on a Hilbert space, then Chebyshev iteration or conjugate-gradient iteration yields significantly better convergence rates [14, 23]. An efficient implementation of Theorems 2 and 4 below will make use of these acceleration techniques.

2. THREE ITERATIVE RECONSTRUCTION ALGORITHMS

It is clear that for a complete reconstruction of a band-limited signal from its irregular samples the signal must be sampled on a sufficiently dense subset of $\mathbb{Z}_N$. We measure the sampling density by the maximal gap length

$$d := \max_{i=1,\dots,r} (n_{i+1} - n_i), \tag{16}$$

where $n_{r+1} = n_1 + N$ by periodicity. Denote the midpoints between the samples by $l_i = [(n_i + n_{i+1})/2]$, $i = 1, \dots, r$, and $l_{r+1} = l_1 + N$, $l_0 = l_r - N$. Here $[q]$ is the largest integer $\leqslant q \in \mathbb{R}$. Notice that both $l_i - n_i \leqslant [d/2]$ and $n_{i+1} - l_i \leqslant [d/2] + 1$ for $i = 1, \dots, r$. Let χ_i be characteristic function of the segment $\{l_{i-1} + 1, l_{i-1} + 2, \dots, l_i\}$.

Then a discrete band-limited signal can be reconstructed by the following algorithm.

THEOREM 1. *Assume that* $0 \leqslant n_1 < \cdots < n_r \leqslant N - 1$ *is a subsequence of* $\{0, 1, \dots, N - 1\}$ *with* $\max_{i=1,\dots,r}(n_{i+1} - n_i) = d$ *and that*

$$2M(2[d/2] + 1) < N. \tag{17}$$

Then $s \in \mathscr{B}_M$ *is completely determined by the nonuniform samples* $s(n_i)$,

$i = 1, \ldots, r$, and can be reconstructed iteratively by the following algorithm:

$$s_0 = P\left(\sum_{i=1}^{r} s(n_i)\chi_i\right), \tag{18}$$

$$s_{n+1} = s_n + P\left(\sum_{i=1}^{r} \{s(n_i) - s_n(n_i)\}\chi_i\right). \tag{19}$$

Furthermore $s = \lim_{n \mapsto \infty} s_n$ converges geometrically with rate

$$\gamma = \sin \frac{\pi M}{N} \Big/ \sin \frac{\pi}{4[d/2] + 2}.$$

Proof. According to Lemma 3 we only need to estimate $\|s - As\|$ for the approximation operator

$$As = P\left(\sum_{i=1}^{r} s(n_i)\chi_i\right). \tag{20}$$

By definition A maps $\mathscr{B}_M$ into itself. For $s \in \mathscr{B}_M$ we have $s = Ps = P(\sum_{i=1}^{r} s\chi_i)$, and we can write the energy norm $s - As$ as follows:

$$\|s - As\|^2 = \left\| P\left(\sum_{i=1}^{r} \{s - s(n_i)\}\chi_i\right)\right\|^2 \leqslant \left\| \sum_{i=1}^{r} [s - s(n_i)]\,\chi_i\right\|^2 \tag{21}$$

$$= \sum_{n=0}^{N-1} \left| \sum_{i=1}^{r} \{s(n) - s(n_i)\}\chi_i(n)\right|^2 = \sum_{i=1}^{r} \sum_{n=l_{i-1}+1}^{l_i} |s(n) - s(n_i)|^2$$

Next we apply (9) of Lemma 2 to each term in the last sum for the sequences $\tilde{s}(n) = s(n_i - 1 + j) - s(n_i)$, $j = 1, \ldots, l_i - n_i + 1$, and to $\tilde{s}(n)$

$= s(n_i + 1 - j) - s(n_i)$, $j = 1, \ldots, n_i - l_{i-1}$; we obtain

$$\sum_{n=n_i}^{l_i} |s(n) - s(n_i)|^2 \leqslant \left(4 \sin^2 \frac{\pi}{2[2(l_i - n_i + 1) - 1]}\right)^{-1} \sum_{n=n_i}^{l_i - 1} |\Delta s(n)|^2$$

and

$$\sum_{n=l_{i-1}+1}^{n_i} |s(n) - s(n_i)|^2 \leqslant \left(4 \sin^2 \frac{\pi}{2[2(n_i - l_{i-1}) - 1]}\right)^{-1} \sum_{n=l_{i-1}+1}^{n_i - 1} |\Delta s(n)|^2.$$

Since $\max(l_i - n_i + 1, n_i - l_{i-1}) \leqslant [d/2] + 1$, both sin terms are smaller than

$$\left(4 \sin^2 \frac{\pi}{4[d/2] + 2}\right)^{-1}.$$

Combining these estimates, we obtain

$$\|s - As\|^2 \leqslant \left(4 \sin^2 \frac{\pi}{4[d/2] + 2}\right)^{-1} \sum_{i=1}^{r} \sum_{n=l_{i-1}+1}^{l_i} |\Delta s(n)|^2$$

$$= \left(4 \sin^2 \frac{\pi}{4[d/2] + 2}\right)^{-1} \|\Delta s\|^2 \tag{22}$$

Since $s \in \mathscr{B}_M$, Bernstein's inequality yields

$$\|s - As\|^2 \leqslant \frac{\sin^2 \dfrac{\pi M}{N}}{\sin^2 \dfrac{\pi}{4[d/2] + 2}} \|s\|^2. \tag{23}$$

Consequently, on the finite-dimensional subspace $\mathscr{B}_M$ we obtain

$$\|\mathrm{Id} - A\|_{\mathrm{Op}} \leqslant \sin \frac{\pi M}{N} \Big/ \sin \frac{\pi}{4[d/2] + 2}.$$

Because sin is monotonic on the interval $[0, \pi/2]$ and $0 < \pi/(4[d/2] + 2)$, $(\pi M/N) < \pi/2$, the operator norm is smaller than one if $\pi M/N < \pi(4[d/2] + 2)$, which is exactly the hypothesis (17). An application of Lemma 3 finishes the proof. ∎

The approximation operator in Theorem 1 can be modified in several ways so that it is easier to handle in numerical implementations or so that it gives a better convergence rate. We will give two variations of Theorem 1.

Let $\varepsilon_l \in l^2(\mathbb{Z}_N)$ be the sequence $\varepsilon_l(l) = 1$ and $\varepsilon_l(k) = 0$ for $k \neq l$. The projections onto $\mathscr{B}_M$ are

$$P\varepsilon_l(j) = \frac{\sin\{(2M + 1)\pi(j - l)/N\}}{N \sin\{\pi(j - l)/N\}}.$$

We keep the notation of Theorem 1, where l_i are the midpoints between the sampling points n_i, and set

$$\Delta_i = l_{i+1} - l_i, \qquad d = \max_{i=1,\dots,r} (n_{i+1} - n_i), \quad \text{and}$$

$$\gamma = \sin \frac{\pi M}{N} \Big/ \sin \frac{\pi}{4[d/2] + 2}.$$

THEOREM 2. *Let $0 \leqslant n_i < \cdots < n_r < N$ be a subsequence of $\{0, 1, \dots, N - 1\}$ with*

$$2M(2[d/2] + 1) < N, \tag{24}$$

and let λ be a relaxation parameter with $0 < \lambda < 2/(1 + \gamma)^2$. Then every $s \in \mathscr{B}_M$ can be reconstructed iteratively from the samples $s(n_i)$, $i = 1, \dots, r$, by

$$s_0 = \lambda \sum_{i=1}^{r} s(n_i) \Delta_i P\varepsilon_{n_i} \tag{25}$$

and

$$s_{n+1} = s_n + \lambda \sum_{i=1}^{r} \{s(n_i) - s_n(n_i)\}\Delta_i P\varepsilon_{n_i}. \tag{26}$$

The algorithm converges to s geometrically with the rate $\tilde{\gamma} = \max\{|1 - \lambda(1 + \gamma^2)|, |1 - \lambda(1 - \gamma)^2|\}$.

Proof. We consider the approximation operator

$$Bs = \lambda \sum_{i=1}^{r} s(n_i)\Delta_i P\varepsilon_{n_i}. \tag{27}$$

To estimate $\|s - Bs\|$, we first rewrite (23): by the assumption (17) we have $\|\mathrm{Id} - A\|_{\mathrm{Op}} < \gamma$ and consequently $\|A^{-1}\|_{\mathrm{Op}} < (1 - \gamma)^{-1}$. For $s \in \mathscr{B}_M$ we obtain

$$\begin{aligned}(1-\gamma)^2\|s\|^2 &= (1-\gamma)^2\|A^{-1}As\|^2 \\ &\leqslant (1-\gamma)^2\|A^{-1}\|_{\mathrm{Op}}^2 \left\| P\left(\sum_{i=1}^{r} s(n_i)\chi_i\right)\right\|^2 \\ &\leqslant \left\|\sum_{i=1}^{r} s(n_i)\chi_i\right\|^2 = \sum_{i=1}^{r} |s(n_i)|^2\Delta_i \end{aligned} \tag{28}$$

as a lower estimate for $\sum_{i=1}^{r}|s(n_i)|^2\Delta_i$, and

$$\sum_{i=1}^{r} |s(n_i)|^2\Delta_i \leqslant \left(\|s\| + \left\|s - \sum_{i=1}^{r} s(n_i)\chi_i\right\|\right)^2 \leqslant (1+\gamma)^2\|s\|^2 \tag{29}$$

as an upper estimate. Now consider

$$\langle(\mathrm{Id} - B)s, s\rangle = \|s\|^2 - \lambda\left\langle \sum_{i=1}^{r} s(n_i)\Delta_i P\varepsilon_{n_i}, s\right\rangle = \|s\|^2 - \lambda\sum_{i=1}^{r} |s(n_i)|^2\Delta_i$$

The inequalities (28) and (29) imply

$$\{1 - \lambda(1+\gamma)^2\}\|s\|^2 \leqslant \langle(\mathrm{Id} - B)s, s\rangle \leqslant \{1 - \lambda(1-\gamma)^2\}\|s\|^2. \quad (30)$$

Thus the spectrum of $\mathrm{Id} - B$ is contained in the interval $[1 - \lambda(1+\gamma)^2, 1 - \lambda(1-\gamma)^2]$, and $\|\mathrm{Id} - B\|_{\mathrm{op}} \leqslant \max\{|1 - \lambda(1+\gamma)^2|, |1 - \lambda(1-\gamma)^2|\} < 1$ for $\lambda \in (0, 2/(1+\gamma)^2)$. ∎

REMARKS.

(1) The estimates (28) and (29) are the discrete analog of the famous frame estimates by Duffin and Schaeffer [6]. The novelty in the discrete case is the explicit estimates for the rate of convergence in terms of the sampling density and the size of the spectrum, together with the use of the weights Δ_i. It is easy to see that the value $\lambda = 2\{(1+\gamma)^2 + (1-\gamma)^2\}^{-1}$ gives the best rate of convergence.

(2) In each iteration step we need to know only the values of Bs at n_i. Writing

$$B_{ki} = \Delta_i \lambda N^{-1}\left(\sin\frac{(2M+1)\pi(n_k - n_i)}{N}\right)\left(\sin\frac{\pi(n_k - n_i)}{N}\right)^{-1}, \qquad i, k = 1, \ldots, r,$$

the iterations amount to the multiplication of the vector $s(n_i)$ by the $r \times r$ matrix B. From the point of view of linear algebra this algorithm is a one-step Richardson method.

Given the data $s(n_i)$, $i = 1, \ldots, r$, on $\mathbb{Z}_N$, let T_1 be piecewise linear interpolation, i.e.,

$$T_1 s(n) = s(n_i) + \frac{s(n_{i+1}) - s(n_i)}{n_{i+1} - n_i}(n - n_i) \qquad \text{for} \quad n_i \leqslant n \leqslant n_{i+1}, \quad (31)$$

and set $A_1 = PT_1$. Then A_1 maps $\mathscr{B}_M$ into $\mathscr{B}_M$.

THEOREM 3. *Assume that* $0 \leqslant n_1 < \cdots < n_r < N$ *is a subsequence of* $\{0, 1, \ldots, N-1\}$ *with* $\max_{i=1,\ldots,r}(n_{i+1} - n_i) = d$ *and that*

$$2Md < N. \quad (32)$$

Then every $s \in \mathscr{B}_M$ can be reconstructed by the following algorithm:

$$s_0 = A_1 s \quad \text{and} \quad s_{n+1} = s_n + A_1(s - s_n) \tag{33}$$

Furthermore, $s = \lim_{n \to \infty} s_n$ converges geometrically with rate

$$\gamma = \frac{\sin^2(\pi M/N)}{\sin^2(\pi/2d)}.$$

Proof. The proof is almost identical with that of Theorem 1, and we only give the necessary modifications. With the abbreviation $\{s(n_{i+1}) - s(n_i)\}/(n_{i+1} - n_i) = \delta s_i$, we compute as in (21)

$$\|s - A_1 s\|^2 = \sum_{i=1}^{r} \sum_{n=n_i}^{n_{i+1}} |s(n) - s(n_i) - \delta s_i(n - n_i)|^2. \tag{34}$$

Next we apply (10) of Lemma 2 to each sequence $\tilde{s}(j) = s(n_i + j) - s(n_i) - \delta s_i j$ for $j = 1, \dots, n_{i+1} - n_i - 1$ $[\tilde{s}(0) = \tilde{s}(n_{i+1} - n_i) = 0]$, and we obtain

$$\begin{aligned} \|s - A_1 s\| &\leqslant \sum_{i=1}^{r} \left(16 \sin^4 \frac{\pi}{2(n_{i+1} - n_i)}\right)^{-1} \sum_{n=n_i+1}^{n_{i+1}-2} |\Delta^2 s(n)|^2 \\ &\leqslant \left(16 \sin^4 \frac{\pi}{2d}\right)^{-1} \|\Delta^2 s\|^2 \leqslant \frac{\sin^4(\pi M/N)}{\sin^4(\pi/2d)} \|s\|^2. \end{aligned} \tag{35}$$

The last inequality follows after employing Bernstein's inequality twice. Consequently $\|\mathrm{Id} - A_1\|_{\mathrm{Op}} \leqslant \gamma < 1$ on $\mathscr{B}_M$ whenever $2Md < N$, and Lemma 3 applies. ■

3. INTERPOLATION OF TRIGONOMETRIC POLYNOMIALS

In the preceding section we treated the irregular sampling problem for discrete band-limited signals in the precise version in which it has been implemented. Now we change the point of view and interpret the algorithms of Theorems 1 and 2 as methods for the interpolation of trigonometric polynomials.

For this we consider the torus group $\mathbb{T}$, i.e., the unit interval $[0, 1)$ with addition modulo one, and for given $M > 0$ the $(2M + 1)$-dimensional space of all trigonometric polynomials of order M, i.e.,

$$\mathscr{P}_M = \left\{ f : f(x) = \sum_{-M}^{M} a_k e^{2\pi i k x} \right\}. \tag{36}$$

If we identify $\mathbb{Z}_N$ with the subgroup $\{j/N,\ j = 0, \dots, N-1\}$ of $\mathbb{T}$, then a band-limited discrete signal $f \in \mathscr{B}_M$, $f(j) = \sum_{|k| \leq M} \hat{f}(k) e^{2\pi i jk/N}$ is just the restriction of the trigonometric polynomial $p(x) = \sum_{|k| \leq M} \hat{f}(k) e^{2\pi i k x} \in \mathscr{P}_M$ to the subgroup $\mathbb{Z}_N$. Our original sampling problem now turns into the following interpolation problem: Given a trigonometric polynomial of known degree M and with values $p(x_j)$, $j = 1, \dots, r$, find $p \in \mathscr{P}_M$. In the context of the previous section the sampling points were of the form $x_j = n_j/N$ for some large N, and Theorems 1–3 provide an iterative reconstruction algorithm and an error estimate.

Of course, in theory there is no problem, since any $p \in \mathscr{P}_M$ is uniquely determined by its values in at least $2M + 1$ distinct points in $\mathbb{T}$, and a reconstruction could be written explicitly as a Lagrange polynomial. However, the practical reconstruction and error estimates are still a nontrivial problem, when the number of sampling values is large, typically $r \cong 10^3$, say, and when the system is overdetermined. Theorems 1–3 give a practical and implementable answer.

For the interpolation of trigonometric polynomials it is not necessary to assume sampling points of the form j/N. We may take any increasing sequence x_k, $0 \leq x_1 < x_2 < \cdots < x_{r-1} < x_r < 1$. We denote the midpoints by $y_j = (x_{j+1} + x_j)/2$ and the weights $w_j = y_j - y_{j-1} = (x_{j+1} - x_{j-1})/2$. To preserve periodicity, we set $x_{r+1} = x_1 + 1$ and $x_0 = x_r - 1$, and we have

$$\sum_{j=1}^{r} w_j = 1. \tag{37}$$

Furthermore, let

$$D_M(x) = \sum_{k=-M}^{M} e^{2\pi i k x} = \frac{\sin(2M + 1)\pi x}{\sin \pi x} \tag{38}$$

denote the Dirichlet kernel. Then the orthogonal projection $P : L^2(\mathbb{T}) \mapsto \mathscr{P}_M$ is given as the convolution $Pf = f * D_M$, and for any $p \in \mathscr{P}_M$ the point evaluations are

$$p(y) = Pp(y) = \langle p, D_M(\cdot - y) \rangle. \tag{39}$$

The following theorem is the analog of Theorem 2.

THEOREM 2. *Let $p \in \mathscr{P}_M$, and x_j, $j = 1, \dots, r$, be an increasing sequence in* $[0, 1)$. *If*

$$\delta := \max_{j=1,\dots,r} (x_{j+1} - x_j) < \frac{1}{2M}, \tag{40}$$

then p is uniquely determined by its values $p(x_j)$. If λ is a relaxation parameter with $0 < \lambda < 2/(1 + 2\delta M)^2$, then p can be reconstructed by the iterative algorithm

$$p_0(x) = \lambda \sum_{j=1}^{r} p(x_j) w_j D_M(x - x_j), \tag{41}$$

$$p_{n+1}(x) = p_n(x) + \lambda \sum_{j=1}^{r} \left[p(x_j) - p_n(x_j) \right] w_j D_M(x - x_j). \tag{42}$$

The sequence p_n converges to p geometrically in the $L^2(\mathbb{T})$ norm with rate $\gamma = \max\{|1 - \lambda(1 + 2\delta M^2)|, |1 - \lambda(1 - 2\delta M)^2|\}$.

REMARKS.

(a) The condition (40) implies that there are at least $2M + 1$ sampling points. Uniqueness is therefore obvious.

Also, with the identifications made above, Theorem 2 is a consequence of Theorem 4. However, since the algorithm (25), (26) and not its continuous analog in Theorem 4 is the basis for the numerical implementation, we prefer to treat the discrete irregular sampling in detail.

(b) Interpolation of trigonometric polynomials at nonuniformly spaced nodes has been treated in several places; e.g. see [15, 21] and the references cited there. In particular, the recent algorithm in [21] seems very accurate and efficient and requires only $O(rM)$ arithmetic operations. It would therefore be interesting to compare the numerical performance of the iterative and the direct algorithms for the irregular-sampling problem.

Proof.

(A) The proof follows exactly the same steps as the proofs of Theorems 1 and 2. We just indicate the modifications and leave the details to the reader. The discrete Wirtinger inequality is replaced by the classical Wirtinger inequality (see [13] or [18]): *If* $f \in L^2(a, b)$ *and either* $f(a) = 0$ *or* $f(b) = 0$, *then*

$$\int_a^b |f(x)|^2\, dx \leqslant \frac{4}{\pi^2}(b-a)^2 \int_a^b |f'(x)|^2\, dx. \tag{43}$$

The L^2 version of Bernstein's inequality for trigonometric polynomial is obvious:

$$\|p'\|_2 \leqslant 2\pi M \|p\|_2 \qquad \text{for all} \quad f \in \mathscr{P}_M \tag{44}$$

(B) Let χ_j be the characteristic function of the interval $[y_{j-1}, y_j]$, and consider

$$\mathscr{A}p = P\left(\sum_{j=1}^{r} p(x_j)\chi_j\right).$$

Then we obtain as in (21)

$$\|p - \mathscr{A}p\|_2^2 \leqslant \sum_{j=1}^{r} \int_{y_{j-1}}^{y_j} |p(x) - p(x_j)|^2\, dx$$

Since $y_j - x_j \leqslant \delta/2$ and $x_j - y_{j-1} \leqslant \delta/2$ by definition, Wirtinger's inequality (43), applied to the integrals $\int_{y_{j-1}}^{x_j}$ and $\int_{x_j}^{y_j}$, implies

$$\sum_{j=1}^{r} \int_{y_{j-1}}^{y_j} |p(x) - p(x_j)|^2\, dx \leqslant \frac{\delta^2}{\pi^2} \sum_{j=1}^{r} \int_{y_{j-1}}^{y_j} |p'(x)|^2\, dx = \frac{\delta^2}{\pi^2}\|p'\|_2^2$$

Because $p \in \mathscr{P}_M$, (44) yields finally

$$\|p - \mathscr{A}p\|_2 \leqslant 2\delta M \|p\|_2. \tag{45}$$

For $2\delta M < 1$, $\mathscr{A}$ is invertible on $\mathscr{P}_M$ with operator norm $\|A^{-1}\|_{\mathrm{Op}} \leqslant (1 - 2\delta M)^{-1}$, and p could be reconstructed from $p(x_j)$ by means of $\mathscr{A}$ and Lemma 3. From (45) we deduce as in (28) and (29)

$$\begin{aligned}(1 - 2\delta M)^2 \|p\|_2^2 &\leqslant \left\| \sum_{j=1}^{r} p(x_j)\chi_j \right\|_2^2 = \sum_{j=1}^{r} |p(x_j)|^2 w_j \\ &\leqslant (1 + 2\delta M)^2 \|p\|_2^2. \end{aligned} \tag{46}$$

(C) Now consider $\mathscr{B} : \mathscr{P}_M \mapsto \mathscr{P}_M$,

$$\mathscr{B}p(x) = \sum_{j=1}^{r} p(x_j) w_j D_M(x - x_j). \tag{47}$$

We observe that by (39)

$$\langle \mathscr{B}p, p \rangle = \sum_{j=1}^{r} |p(x_j)|^2 w_j.$$

Combined with (46) and the hypothesis (40) this means that $\mathscr{B}$ is invertible on $\mathscr{P}_M$ and that

$$\|\mathrm{Id} - \mathscr{B}\|_{\mathrm{Op}} \leqslant \max\left\{ \left|1 - \lambda(1 + 2\delta M)^2\right|, \left|1 - \lambda(1 - 2\delta M)^2\right| \right\}.$$

If $0 < \lambda < 2/(1 + 2\delta M)^2$, then $\|\mathrm{Id} - \mathscr{B}\|_{\mathrm{Op}} < 1$ and the iterative algorithm follows again from Lemma 3. ■

4. AN EQUIVALENT ALGORITHM USING TOEPLITZ MATRICES

Instead of running the algorithm on the space $\mathscr{P}_M$ of trigonometric polynomials, one may formulate it as a procedure acting on the coefficients of these polynomials.

For this we compute the action of $\mathscr{B}p(x) = \sum_{j=1}^{r} p(x_j) w_j D_M(x - x_j)$ on a trigonometric polynomial $p(x) = \sum_{k=-M}^{M} a_k e^{2\pi i k x} \in \mathscr{P}_M$:

$$\begin{aligned} \mathscr{B}p(x) &= \sum_{j=1}^{r} \left(\sum_{k=-M}^{M} a_k e^{2\pi i k x_j} \right) w_j \left(\sum_{l=-M}^{M} e^{2\pi i l (x - x_j)} \right) \\ &= \sum_{l=-M}^{M} \left[\sum_{k=-M}^{M} \left(\sum_{j=1}^{r} w_j e^{-2\pi i x_j (l-k)} \right) a_k \right] e^{2\pi i l x}. \end{aligned}$$

With the notation

$$C_{lk} = \sum_{j=1}^{r} w_j e^{-2\pi i x_j (l-k)}, \tag{48}$$

the coefficients $\mathbf{b} = (b_k)_{|k| \leqslant M}$ of the polynomial $\mathscr{B}p \in \mathscr{P}_M$ are obtained from the coefficients $\mathbf{a} = (a_k)_{|k| \leqslant M}$ of p by matrix multiplication,

$$\mathbf{b} = C\mathbf{a}. \tag{49}$$

Since $\langle p, \mathscr{B}p \rangle = \sum_{j=1}^{r} w_j |p(x_j)|^2 \geqslant 0$ for $p \in \mathscr{P}_M$, C is a positive definite $(2M+1) \times (2M+1)$ Toeplitz matrix which is defined by the values

$$\gamma_l = \sum_{j=1}^{r} w_j e^{-2\pi i x_j l} \qquad \text{for} \quad |l| \leqslant 2M \tag{50}$$

and $C_{l,k} = \gamma_{l-k}$ for $|k|, |l| \leqslant M$. By the definition of the weights w_j we have $\gamma_0 = 1$.

Finally, we compute the first step of the iteration:

$$p_0(x) = \sum_{j=1}^{r} p(x_j) w_j D_M(x - x_j) = \sum_{l=-M}^{M} \left(\sum_{j=1}^{r} p(x_j) w_j e^{-2\pi i x_j l} \right) e^{2\pi i x l}.$$

The algorithm of Theorem 4 can now be formulated as follows.

THEOREM 5. *Let $p \in \mathscr{P}_M$ and $0 \leqslant x_1 < x_2 < \cdots < x_r < 1$ be given. If*

$$\delta := \max_{j=1,\dots,r} (x_{j+1} - x_j) < \frac{1}{2M}, \tag{51}$$

then C is invertible and has a condition number

$$\operatorname{cond} C \leqslant \left(\frac{1 + 2\delta M}{1 - 2\delta M} \right)^2. \tag{52}$$

Any $p \in \mathscr{P}_M$ can be reconstructed from its values $p(x_j)$, $j = 1, \dots, r$ as follows: Set $\mathbf{a}^{(0)} = (a_l^{(0)})_{|l| \leqslant M}$, where

$$a_l^{(0)} = \sum_{j=1}^{r} p(x_j) w_j e^{-2\pi i x_j l}; \tag{53}$$

then $\mathbf{a} = (a_l)_{|l| \leqslant M} = C^{-1}\mathbf{a}^{(0)}$ and $p(x) = \sum_{l=-M}^{M} a_l e^{2\pi i l x}$. Given a relaxation *parameter* λ, $0 < \lambda < 2/(1 + 2\delta M)^2$, *the solution* $\mathbf{a}$ *can be approximated iteratively by*

$$\mathbf{a}^{(n+1)} = \mathbf{a}^{(n)} + \lambda C(\mathbf{a}^{(0)} - \mathbf{a}^{(n)})$$

with $\lim_{n \mapsto \infty} \mathbf{a}^{(n)} = \mathbf{a}$.

In principle, the Toeplitz matrix C contains the complete solution of the finite-dimensional irregular-sampling problem for band-limited signals and trigonometric polynomial. Recall that by Carathéodory's theorem [11] the entries γ_l of an positive-definite Toeplitz matrix are always of the form $\gamma_l = \sum_{j=1}^{r} w_j e^{-2\pi i x_j l}$ for $|l| \leqslant N$ for some $w_j > 0$ and $x_j \in [0, 1)$. The following simple criterion for the invertibility is implicit in Carathéodory's theorem. We state it for comparison with Theorem 5.

PROPOSITION 1. *Given* $M > 0$, $M \in \mathbb{N}$, *and a sampling set* $0 \leqslant x_1 < x_2 < \cdots < x_r < 1$. *Then the following are equivalent*:

(1) *Every* $p \in \mathscr{P}_M$ *is uniquely determined by its values* $p(x_j)$, $j = 1, \ldots, r$.

(2) *The cardinality* r *of the sampling set* (x_j) *is at least* $2M + 1$.

(3) C *is invertible.*

(4) *Let* $d_j > 0$, $j = 1, \ldots, r$ *be arbitrary, and set* $\tau_l = \sum_{j=1}^{r} d_j e^{2\pi i x_j l}$. *Then the self-adjoint Toeplitz matrix* $T_{l,k} = \tau_{l-k}$, $|l|, |k| \leqslant M$, *is invertible.*

Proof. (1) and (2) are equivalent because the ordinary polynomial $Q(z) = \sum_{k=0}^{2M} a_{k-M} z^k$ has exactly $2M$ zeros, and therefore

$$p(x) = \sum_{k=-M}^{M} a_k e^{2\pi i k x} = e^{-2\pi i M x} Q(e^{2\pi i x}) \in \mathscr{P}_M$$

is determined by its values at any $2M + 1$ points.

(2) $\Rightarrow$ (4): If T is not invertible, then there is a sequence $\mathbf{a} = (a_k)_{|k| \leqslant M} \neq 0$ and a nonzero polynomial $p(x) = \sum_{k=-M}^{M} a_k e^{2\pi i k x}$ such that $Ta = 0$. In particular,

$$0 = \langle Ta, a \rangle = \sum_{k,l=-M}^{M} \tau_{l-k} a_k \bar{a}_l$$

$$= \sum_{k,l=-M}^{M} \sum_{j=1}^{r} d_j e^{2\pi i x_j (k-l)} a_k \bar{a}_l = \sum_{j=1}^{r} d_j |p(x_j)|^2.$$

Now $p(x_j) = 0$ for $j = 1, \ldots, r$ and $p \in \mathscr{P}_M$, $p \neq 0$ entails $r \leqslant 2M$.

(4) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): Let $p(x) = \sum_{k=-M}^{M} a_k e^{2\pi i k x} \in \mathscr{P}_M$, and assume that the values $p(x_j)$, $j = 1, \ldots, r$, are given. By (53) the initial step of the reconstruction algorithm is

$$a_l^{(0)} = \lambda \sum_{j=1}^{r} p(x_j) w_j e^{-2\pi i x_j l} = (C\mathbf{a})_l,$$

and we can compute $\mathbf{a} = C^{-1}\mathbf{a}^{(0)}$. Thus p is uniquely determined. ■

REMARKS. Although Theorem 5 is just a different formulation of Theorems 2 and 4, it is of interest for several reasons.

(1) Under the assumptions of Theorem 5, the condition number of C is reasonably small. Therefore it makes sense to solve the reconstruction problem directly by matrix inversion, an undertaking that seems to be rather hopeless in the context of Theorems 1–4. Since the inversion of Toeplitz matrices is a standard problem in applications, many fast algorithms are known. See [17, 16, 1, 22, 5] for some recent contributions and a wealth of references. Some of the "superfast" inversion algorithms require only $O(N \log^2 N)$ or even $O(N \log N)$ operations. Compared to the original system of equations (5), this would reduce the amount of computation considerably.

(2) Equation (52) is a worst-case estimate for the condition number of C. It depends only on M and on the maximal gap between the samples; it does not reflect any geometric peculiarities of the distribution of the sampling set. In our numerical experiments the condition numbers were in most cases much smaller than predicted by (52).

(3) It is worth emphasizing that the use of the weights w_j is crucial for good estimates. It is easy to see that in the unweighted case, i.e. $w_j = 1$, the upper bound in (46) can be arbitrarily large, if the sampling set consists of several dense clusters of points.

(4) If the sampling set $\{x_j,\ j = 1, \ldots, r\}$ is fixed and the degree M is varied, then one can observe the following qualitative behavior for the extremal eigenvalues of the matrix $C_{l,k}^M = \gamma_{l-k}$, $|k|, |l| \leqslant M$: The largest eigenvalue of C^M increases slowly with increasing M, and the smallest eigenvalue decreases slowly, as long as the condition (51) is approximately satisfied — $\delta M < 2$, say. However, as $2M + 1 \mapsto r$, the smallest eigenvalue converges fast to 0, and for $2M + 1 \approx r$ one can find eigenvalues of size 10^{-12} and smaller. Without further conditions on the sampling set, the proposition seems to be of little practical values.

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